INVESTIGATION OF THE STABILITY OF THE SOLUTION OF A LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH PERIODIC COEFFICIENTS

(ISSLEDOVANIE USTOICHIVOSTI RESHENIIA LINEINOGO DIFFERENTSIAL'NOGO URAVNENIIA VTOROGO PORIADKA S PERIODICHESKIMI KOEFFITSENTAMI)

PMM Vol.22, No.2, 1958, pp.250-253

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(Received 2 July 1958)

1. In article [1] a method is given for the determination of the eigenvalues of the solution of differential equations of the second order with periodic coefficients. We shall review in brief the method. Let it be given a differential equation

$$\frac{d^2x}{d\varphi^2} + A\frac{dx}{d\varphi} + Bx = 0 \tag{1.1}$$

whose coefficients A and B are periodic functions of ϕ with a period 2π. If the coefficient A has a derivative with respect to ϕ, then the equation (1.1) is reducible to the equation of Hill's type

$$\frac{d^2z}{d\varphi^2} + Kz = 0 \tag{1.2}$$

The coefficient K is also a periodic function of ϕ with a period 2π. Let K depend also on a small parameter μ, such that at $\mu = 0$, the function $K(\phi, \mu)$ is a constant. Under certain restrictions [1, 2] the function $K(\phi, \mu)$ can be represented in the form of a series

$$K(\varphi, \mu) = a + \sum_{n=1}^{\infty}\left[p_{0n} + \sum_{m=1}^{\infty}(p_{mn}\cos m\varphi + q_{mn}\sin m\varphi)\right]\mu^n \tag{1.3}$$

The solution of (1.2) is to be obtained in the following form

$$z = e^{i\nu\varphi}\sum_{k=-\infty}^{\infty} H_k e^{ik\varphi}$$

The magnitude ν is to be found from the equation

$$\sin^2\pi\nu = \pi^2 D(\mu) \tag{1.4}$$

where $D(\mu)$ is an infinite determinant, the elements of which depend on the coefficients a, p_{mn} and q_{mn} of the series (1.3). This determinant can itself be represented in the form of a series

$$D(\mu) = D_0 + D_1\mu + D_2\mu^2 + \ldots \tag{1.5}$$

Reprint Order No. PMM 28.

The smallest eigenvalue of the solution of the equation (1.1) is determined from the formula

$$\chi = \frac{1}{2} A_0 - |\operatorname{Im} \nu| \tag{1.6}$$

where A_0 is the constant in the Fourier series of the expansion of $A(\phi)$.

Let the coefficients A and B of the equation (1.1) be dependent also upon some parameter a of any arbitrary magnitude different from μ (or of several parameters). Then the determinant D and the eigenvalue χ will also depend on this parameter.

From the expression (1.6) it follows that the maximum possible stability of the system at a given value of the parameter a is obtained when $\operatorname{Im} \nu = 0$. In this case the eigenvalue $\chi = \frac{1}{2} A_0$. But the magnitude $\operatorname{Im} \nu$ is equal to zero if

$$0 \leqslant D \leqslant \frac{1}{\pi^2} \tag{1.7}$$

and is different from zero, if D is outside of this range.

If $A_0 = 0$ the regions where the condition (1.7) is not satisfied are regions of instability. If $A_0 = 0$, at points where $D_0 = 0$ and $D_0 = \pi^{-2}$ instability may also result in view of the resonance with external forces. If $A_0 > 0$, the regions where condition (1.7) is not fulfilled are characterized by a decrease of stability as compared with the maximum possible one obtainable.

In order to obtain a complete picture of the change of the eigenvalues as a function of the change of parameters, considering only a minimum number of points, it is necessary to determine beforehand the limits where condition (1.7) is satisfied. For this purpose let us consider the first member of the expansion (1.5) as a function of the parameter a

$$D_0 = \frac{\sin^2 \pi \sqrt{a}}{\pi^2} \tag{1.8}$$

Since the magnitude a, dependent on a, may take any real values, the coefficient D_0 may change from $-\infty$ to $1/\pi^2$. Let us now determine all the points of the curve D_0 as functions of a (Fig. 1) which lie on the border of the strip (1.7). These points are found from the equation

$$a = \frac{1}{4} n^2 \qquad (n = 0, 1, 2, \ldots) \tag{1.9}$$

At $n = 0$, and at even values of n, we have $D_0 = 0$, and at odd values of n, $D_0 = \pi^{-2}$. Note that all the intersection points of the curve D_0 with the lower boundary of the strip (1.7) correspond to $n = 0$.

With the increase of the parameter μ from zero, the curve $D(\mu)$, which coincides with the curve D_0 at $\mu = 0$, will be deformed continuously and in the neighborhood of those points which lie on the boundary of the strip (1.7) but are not the points of intersection of the curve D_0 with the lower boundary of the strip, may lie beyond the boundary of the strip (1.7). On the graph χ as a function of a in the neighborhood of the corresponding points there will appear new branches below the curve $\chi = \frac{1}{2}A_0$. Those points on the curve $\chi = \frac{1}{2}A_0$ which serve for the formation of the branches due to the increase of the parameter μ, we shall call critical points of the first kind. Thus to each critical point of the first kind on the graph $(\chi - a)$ there corresponds a one-parameter family of curvilinear branches with a parameter equal to μ. Note that critical points of the first kind may appear not only at $\mu = 0$, but also at any value of this parameter.

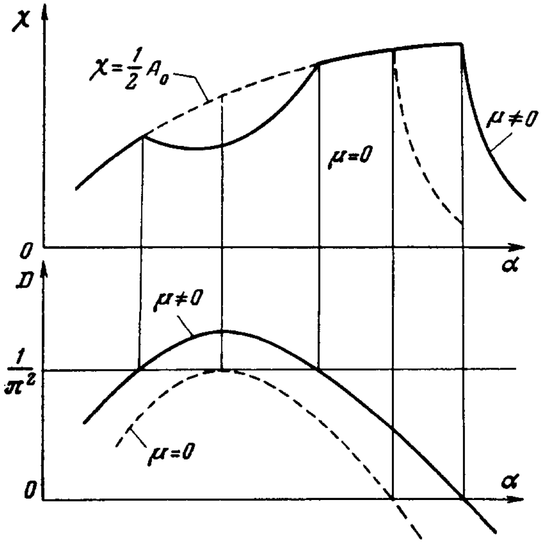

Fig. 1.

Those points on the curve $\chi(a)$, which for $\mu = 0$ are the origin or the end of the curvilinear branches which do not coincide with the curve $\chi = \frac{1}{2}A_0$, we shall call critical points of the second kind. The points of intersection of the curve D_0 with the lower boundary of the strip correspond to these points. With the change of the parameter μ, the above mentioned curvilinear branches χ will deform continually, forming a one-parameter family of curves. To each such family there correspond one or two critical points of the second kind which are the end-points of the curve of the given family at $\mu = 0$.

Let $\nu = \nu_1 + i\nu_2$. It can be easily shown that $\nu_1 = m$, if $D \leqslant 0$, and $\nu_1 = m + \frac{1}{2}$ if $D \geqslant \pi^{-2}$, where m is an integer or zero. Within the strip (1.7), the quantity ν_1 changes continually between the values of the boundary points of the strip.

The solution of equation (1.1) can be represented in the form

$$x(\varphi) = f_0(\varphi)\, f_1(\varphi)\, f_2(\varphi)$$

where $f_0(\phi)$ is an exponential function, $f_1(\phi)$ a periodic function with a period 2π, and $f_2(\phi)$ a periodic function with a period $2\pi/\nu_1$. If $\nu_1 = k/n$, where k and n are positive integers, then the product $f_1(\phi)f_2(\phi)$ is a periodic function with a period $2\pi n$. In this case we are dealing with damped or increasing vibrations. By analogy with the case when the equation has constant coefficients, we shall call the period of this product the period of vibration.

Thus for $D \geqslant \pi^{-2}$ the period is 4π, and for $D \leqslant 0$ the period is 2π, or there occurs an aperiodic motion.

2. Let us take as an example the equation of motion of a propeller blade, which is considered in the articles [1, 2]. This equation is

$$d^2\beta / d\psi^2 + \gamma\left(\tfrac{1}{4} + \tfrac{1}{3}\mu\sin\psi\right) d\beta / d\psi + \left[1 + \gamma\left(\tfrac{1}{3} + \tfrac{1}{2}\mu\sin\psi\right)\mu\cos\psi + \right.$$
$$\left. + \gamma h\left(\tfrac{1}{4} + \tfrac{2}{3}\mu\sin\psi + \tfrac{1}{2}\mu^2\sin^2\psi\right)\right]\beta = F(\psi) \qquad (2.1)$$

where β is the angle of attack, ψ the pitch angle of blade, and μ, γ and h are parameters* (μ is a small parameter). The right hand side of this equation is a periodic function of ϕ with period 2π.

Let us take the equation (2.1) without the right hand side and let us reduce it to the form (1.2). The coefficient K is equal to

$$K(\psi, \mu) = a + (p_{11}\cos\psi + q_{11}\sin\psi)\mu + (p_{02} + p_{22}\cos 2\psi + q_{22}\sin 2\psi)\mu^2$$

Here a, p_{mn}, q_{mn} are determined by the formulas

$$a = 1 - \tfrac{1}{64}\gamma^2 + \tfrac{1}{4}\gamma h, \qquad p_{11} = \tfrac{1}{6}\gamma, \qquad q_{11} = \tfrac{2}{3}\gamma h - \tfrac{1}{24}\gamma^2$$
$$p_{02} = \tfrac{1}{4}\gamma h - \tfrac{1}{72}\gamma^2, \qquad p_{22} = -p_{02}, \qquad q_{22} = \tfrac{1}{4}\gamma$$

The determinant $D(\mu)$ in this case is an even function of μ. The coefficients of the expansion of this determinant D_2 and D_4 are established by the formulas

$$D_2 = \frac{\sin 2\pi\sqrt{a}}{2\pi\sqrt{a}}\left[p_{02} + \frac{p_{11}^2 + q_{11}^2}{2(1-4a)}\right]$$

$$D_4 = \frac{\sin 2\pi\sqrt{a}}{2\pi\sqrt{a}}\left[\frac{p_{22}^2 + q_{22}^2}{8(1-a)} + \frac{3}{8}\,\frac{p_{22}(p_{11}^2 - q_{11}^2) + 2p_{11}q_{11}q_{22}}{(1-a)(1-4a)} + \right.$$
$$+ \frac{1}{2}p_{02}^2\left(\frac{\pi}{\sqrt{a}}\operatorname{ctg} 2\pi\sqrt{a} - \frac{1}{2a}\right) + \frac{p_{02}(p_{11}^2 + q_{11}^2)}{2(1-4a)}\left(\frac{\pi}{\sqrt{a}}\operatorname{ctg} 2\pi\sqrt{a} - \frac{1}{2a} + \frac{4}{1-4a}\right) +$$
$$\left. + \frac{(p_{11}^2 + q_{11}^2)^2}{8(1-4a)^2}\left(\frac{\pi}{\sqrt{a}}\operatorname{ctg} 2\pi\sqrt{a} - \frac{1}{2a} + \frac{4}{1-4a} + \frac{9}{4(1-a)}\right)\right]$$

* All steps for the determination of these parameters are given in the articles [1, 2].

The smallest eigenvalue of the system in this case is

$$\chi = \frac{1}{8}\gamma - |\operatorname{Im}\nu| \tag{2.2}$$

All critical points are on the line $\chi = 1/8\gamma$ which correspond to the maximum possible stability of the blade at the given value of the parameter γ. To obtain the critical points let us consider equation (1.9). This equation connects the two parameters γ and h. Solving it for γ we get

$$\gamma = 8\left(h \pm \sqrt{h^2 + 1 - \tfrac{1}{4}n^2}\right) \qquad (n = 0, 1, 2, \ldots)$$

The parameters γ and h shall be considered in the following ranges

$$0 < \gamma \leqslant 10, \qquad -0.5 \leqslant h \leqslant 1.0$$

In these ranges the changes of these parameters may have three critical points for given values of h

$$\gamma_0 = 8(h + \sqrt{h^2 + 1}), \qquad \gamma_1 = 8\left(h + \sqrt{h^2 + \tfrac{3}{4}}\right), \qquad \gamma_2 = 16h \tag{2.3}$$

The first point is a critical point of the second kind, the other two points, (when using the sign which follows D_0 and the coefficient of D_n, which is not equal to zero), are critical points of the first kind. Note that for the third point $D_2 = 0$. At the critical points the frequencies ν_1 are 0, 1/2, 1 respectively.

Fig. 2 shows the curves D as functions of the parameter γ for $\mu = 0$, for small μ, and for several values of parameter h. On the curve D_0 for $\chi = 8h$ there is a maximum within the strip (1.7). Near this point new critical points of the first order may appear at some arbitrary value of μ.

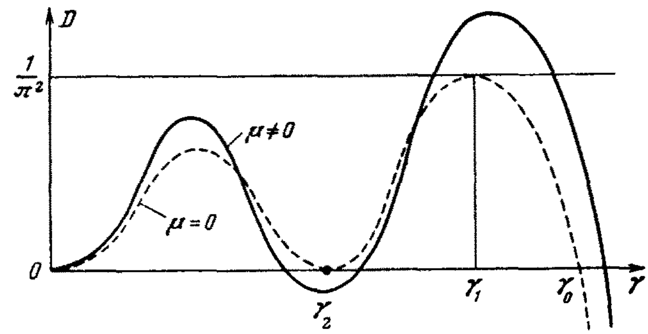

Fig. 2.

Fig. 3 shows curves of the smallest eignevalues established for the parameter γ, with μ from 0 to 0.5. Each graph corresponds to one of four values of the parameter $h = -0.5$, 0, 0.5, and 1.0.

Let us analyze how these curves of the eigenvalues are changing as the parameter h is changing from -0.5 to 1.0. At $h = -0.5$ there are two families of curvilinear branches with critical points $\gamma_0 = 4(\sqrt{5} - 1)$

and $\gamma_1 = 4$. With the increase of the parameter h, these points are moving to the right, and at $h = 0$, their abscissas are becoming $\gamma_0 = 8$ and $\gamma_1 = 4\sqrt{3}$. With further increase of h, these points, together with their corresponding families of curvilinear branches, come to lie beyond the limit of values of γ of practical interest. For instance, at $h = 0.5$ the abscissas of these points are equal to $\gamma_0 = 4(\sqrt{5} + 1)$ and $\gamma_1 = 12$.

At small values of the parameter h to the right of point $\gamma = 0$, there is a new critical point γ_2, which is moving to the right and reaches at $h = 0.5$ an abscissa $\gamma_2 = 8$. At $h = 1.0$ this point together with its family of curvilinear branches, moves over to the domain $\gamma > 10$. Thus for $h = 1.0$ there are no curvilinear branches in the entire range of parameters μ which are of practical interest. To obtain a new critical point corresponding to the above mentioned maximum within the strip (1.7), the value of $\mu = 0.5$ appears to be insufficient.

Note that curvilinear branches of a family which has a critical point γ_2, are closer to a line $\chi = 1/8\gamma$ than the corresponding branches of a family with a critical point γ_1. This is due to the fact that in the expansion $D(\mu)$, the coefficient D_2 is equal to zero for $\gamma = \gamma_2$.

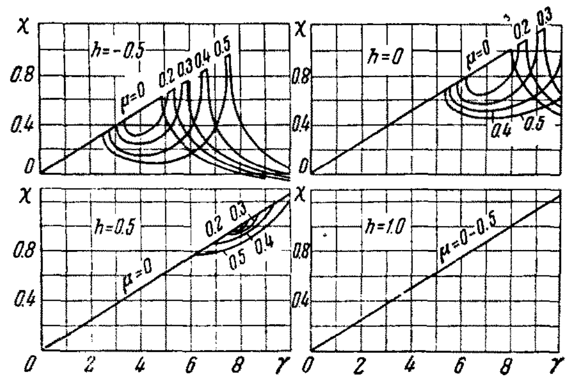

Fig. 3.

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Translated by M.M.